# Nonlinear Evolution of the $m = n = 1$ Kink-Mode in a Periodic $z$-Pinch

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*Dedicated to Professor Dieter Pfirsch on his 60th Birthday*

A previously developed general theory for the nonlinear evolution of external ideal MHD modes is applied to the special case of the $m = n = 1$ kink-mode in $z$-pinch equilibria with constant profile of the safety factor $q$. The Kruskal-Shafranov stability boundary $q = -1$ eludes application of this theory since the kink-mode degenerates there. At the second stability boundary of the mode, stabilization as well as destabilization become possible depending on which values of the aspect ratio and the distance of a stabilizing wall are given.

## Introduction

It is known that in the vicinity of critical points separating regimes of stable and unstable behaviour, the nonlinear dynamics of complex systems can be reduced to simple normal forms under rather general circumstances. However, it is a quite familiar experience that methods which are considered standard in other fields encounter unexpected difficulties when applied to plasmas. For example, linear stability is usually considered as a managable and well understood problem. Nevertheless, the occurrence of degeneracies, singularities, the intermixing of continuous and point spectra and other peculiarities have caused major difficulties in the development of the linear stability theory of plasmas, even in the case of ideal magnetohydrodynamics. Through the use of powerful computers this has finally become a field which is fairly well understood. It can be expected on these grounds that the derivation of normal forms for the nonlinear behaviour of plasma instabilities and their application to specific cases is not a trivial task, even if it is restricted to ideal MHD.

A general theory for the nonlinear evolution of ideal MHD modes was developed in [1] for internal and in [2] for external modes. Although the normal form equations derived there contain only one scalar coefficient for describing the nonlinear behaviour, the procedure one must go through until this is obtained requires a tremendous number of nontrivial steps, especially in the case of external modes which involve the nonlinear motion of the plasma-vacuum boundary. Demonstration of the feasibility of this procedure must therefore be considered as an important step for proofing the usefulness of the general developments in [1] and [2].

In this paper we apply this general theory to the special case of kink-modes in a screw-pinch with circular cross-section and constant profile of the safety-factor $q$ which is surrounded by a vacuum field and a concentric superconducting wall. Periodicity conditions are imposed on perturbations in order to provide topological equivalence with toroidal devices. We consider only the $m = n = 1$ kink-mode since $|m| = |n| = 1$ are the most dangerous cases and other signs of $m$ and $n$ are essentially equivalent. Unfortunately, the difficulties with plasma physics mentioned above come up at the important stability boundary $q = -1$ (Kruskal-Shafranov limit). There, the $m = n = 1$ mode degenerates such that the nonlinear treatment of [2] becomes impossible. We must therefore restrict our consideration to the second boundary of this mode.

In Sect. 1 we shall briefly describe the equilibrium properties of the system under consideration. Section 2 summarizes known results on linear stability which are relevant for our purposes and discusses the degeneracy at $q = -1$. According to the general nonlinear theory of [1] and [2] one must solve the first and second order equations of a hier-

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archy of equations in order to be able to calculate the nonlinear coefficient of the normal form equation. These solutions are derived in Sect. 3. The steps required for calculating the nonlinear coefficient from these solutions turned out extremely lengthy. In order to make this paper readable we have suppressed them and give only reference to a more extended report [3]. A discussion of our main results in physical terms is finally given in Sect. 4.

## 1. The $z$-Pinch Equilibrium

We consider a $z$-pinch of circular cross-section which is surrounded by a vacuum region and a concentric superconducting wall (Fig. 1). Using cylindrical coordinates $r$, $\varphi$, $z$, the magnetic field $\boldsymbol{B}_0$ and $\boldsymbol{B}_0^{\mathrm{v}}$ in the plasma and vacuum region, the current density $\boldsymbol{j}_0$ and the plasma pressure $p_0$ are given by

$$\boldsymbol{B}_0 = B_{0\varphi}(x\boldsymbol{e}_\varphi + b\boldsymbol{e}_z) \quad , \quad \boldsymbol{B}_0^{\mathrm{v}} = B_{0\varphi}(x^{-1}\boldsymbol{e}_\varphi + b\boldsymbol{e}_z)\,,$$
$$\boldsymbol{j}_0 = \left(\frac{2B_{0\varphi}}{r_0}\right)\boldsymbol{e}_z\,, \qquad p_0 = B_{0\varphi}^2(1-x^2)\,, \tag{1.1}$$

where

$$x = r/r_0\,. \tag{1.2}$$

$r_0$ is the equilibrium position of the plasma-vacuum boundary and $B_{0\varphi}$ is the poloidal magnetic field

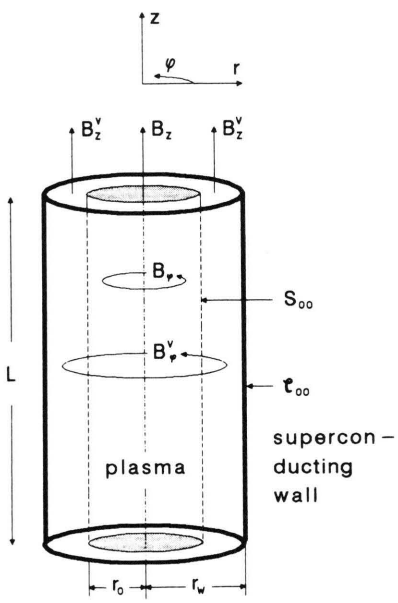

Fig. 1. $z$-pinch equilibrium.

there. The position of the superconducting wall bounding the vacuum region is $r = r_{\mathrm{w}} \geqslant r_0$.

Concerning perturbations from equilibrium we assume periodicity in the $z$-direction with a periodicity length $\Delta z = L$. This allows us to consider the $z$-pinch as a topological tokamak with inverse aspect ratio and safety factor

$$\varkappa = 2\pi r_0/L\,, \quad q = \varkappa b\,, \tag{1.3}$$

respectively.

## 2. Summary of Linear Stability Results

The linear stability problem for the configuration considered was solved in 1957 by Tayler [4, 5] under the special assumption that the plasma perturbations be incompressible. Although we do not want to introduce this assumption we can nevertheless make use of Tayler's results. This is possible because according to the general theory presented in [1] and [2], from the linear theory we need only the marginal mode which is in fact incompressible in the case of kink-modes. Setting

$$\begin{aligned}\tilde{\xi}(\boldsymbol{r},t) &= \xi^{m,k}(x)\exp(im\varphi + ikz + \gamma t)\,,\\ \tilde{\mathfrak{A}}(\boldsymbol{r},t) &= \mathfrak{A}^{m,k}(x)\exp(im\varphi + ikz + \gamma t)\end{aligned} \tag{2.1}$$

for the linear plasma displacement and the vector potential of the perturbational magnetic field in the vacuum region ($\delta\tilde{\boldsymbol{B}}^{\mathrm{v}} = \operatorname{curl}\mathfrak{A}$) respectively (the tilde being supposed to indicate time dependence), Tayler's results for the components of the eigenmodes can be written as

$$\begin{aligned}\xi_r^{m,k}(y) &= C\{J'_{|m|}(y) + m\beta J_{|m|}(y)/y\}\,,\\ \xi_\varphi^{m,k}(y) &= iC\{\beta J'_{|m|}(y) + mJ_{|m|}(y)/y\}\,,\\ \xi_z^{m,k}(y) &= -iC(\beta^2-1)^{1/2}J_{|m|}(y)\,;\\ \mathfrak{A}_z^{m,k}(\hat{\varkappa},x) &= B_{0\varphi}\,\xi_r^{m,k}|_{S_0}F_{|m|}(\hat{\varkappa},x)\,,\\ \mathfrak{A}_\varphi^{m,k}(\hat{\varkappa},x) &= \frac{m}{\hat{\varkappa}x}\mathfrak{A}_z^{m,k}(\hat{\varkappa},x) + (i\Psi_1^{m,k}(\hat{\varkappa},x))'\,,\\ \mathfrak{A}_r^{m,k}(\hat{\varkappa},x) &= \frac{m}{\hat{\varkappa}x}\Psi^{m,k}(\hat{\varkappa},x) - (i\mathfrak{A}_z^{m,k}(\hat{\varkappa},x))'\,.\end{aligned} \tag{2.2}$$

In this, the following definitions are used:

$$y(x) := (\beta^2-1)^{1/2}\hat{\varkappa}x \quad \text{a)}\,, \qquad \beta := \frac{2M}{\Omega^2+M^2} \quad \text{b)}\,,$$

$$M := m + nq \quad \text{c)}\,, \qquad \Omega^2 := \frac{\gamma^2\varrho_0 r_0^2}{B_{0\varphi}^2} \quad \text{d)}\,,$$

$$\hat{\varkappa} := n\varkappa := kr_0 \quad \text{e)}\,, \tag{2.3}$$

$$\psi^{m,k}(\hat{\varkappa},x) = \frac{iMB_{0\varphi}}{\hat{\varkappa}}\,\xi_r^{m,k}|_{S_0}\,G_{|m|}(\hat{\varkappa},x)\,,$$

$$G_m(\hat{\varkappa},x) := \frac{I_m(|\hat{\varkappa}|x)K'_m(|\hat{\varkappa}|x_{\mathrm{w}}) - K_m(|\hat{\varkappa}|x)I'_m(|\hat{\varkappa}|x_{\mathrm{w}})}{I'_m(|\hat{\varkappa}|)K'_m(|\hat{\varkappa}|x_{\mathrm{w}}) - K'_m(|\hat{\varkappa}|)I'_m(|\hat{\varkappa}|x_{\mathrm{w}})}\,,$$

$$F_m(\hat{\varkappa},x) := \frac{I_m(|\hat{\varkappa}|x)K_m(|\hat{\varkappa}|x_{\mathrm{w}}) - K_m(|\hat{\varkappa}|x)I_m(|\hat{\varkappa}|x_{\mathrm{w}})}{I_m(|\hat{\varkappa}|)K_m(|\hat{\varkappa}|x_{\mathrm{w}}) - K_m(|\hat{\varkappa}|)I_m(|\hat{\varkappa}|x_{\mathrm{w}})}\,,$$

$$G'_m(\hat{\varkappa},x) := \partial G_m/\partial(|\hat{\varkappa}|x)\,,$$

$$F'_m(\hat{\varkappa},x) := \partial F_m/\partial(|\hat{\varkappa}|x)\,, \qquad x_{\mathrm{w}} = r_{\mathrm{w}}/r_0\,,$$

$J_{|m|}$, being Bessel-functions and $I_m$, $K_m$ modified Bessel-functions. A boundary condition for the pressure perturbation at $x = 1$ yields the equation

$$\frac{yJ'_{|m|}(y)}{mJ_{|m|}(y)} + \frac{m}{|m|}(y^2+\hat{\varkappa}^2)^{1/2} \cdot \left\{\frac{1}{|\hat{\varkappa}|} + \frac{2}{mMG_{|m|}(\hat{\varkappa},1)} \cdot \frac{y^2}{y^2+\hat{\varkappa}^2}\right\} = 0 \tag{2.4}$$

for $y(1)$. Denoting its solution by $\bar{y}$ and eliminating $\beta$ and $M$ from the first three equations of (2.3) we obtain the dispersion relation

$$\begin{aligned} \Omega^2 &= M\left\{2|\hat{\varkappa}|(\bar{y}^2+\hat{\varkappa}^2)^{-1/2}\frac{m}{|m|} - M\right\} \\ &= (m+nq)\left\{2|\hat{\varkappa}|(\bar{y}^2+\hat{\varkappa}^2)^{-1/2}\frac{m}{|m|} - (m+nq)\right\}. \end{aligned} \tag{2.5}$$

From this, for a mode with given $m$ and $n$ we obtain the boundary of marginal stability by setting $y = 0$. It is easily found that the $|m| = |n| = 1$ modes yield the most stringent marginality conditions on $q$ and are therefore the most dangerous ones. Figures 2a and 2b show the pertinent stability boundaries in a $q$, $x_{\mathrm{w}}^{-2}$-plane for $m = n = \pm 1$ and $m = -n = \pm 1$ respectively. Corresponding to the two brackets in (2.5) which can vanish separately we have two stability boundaries. One of them is independent of the parameters of the problem and is given by $q = -1$ (for $m = n = \pm 1$) or $q = 1$ (for $m = -n = \pm 1$). The other one depends on $\varkappa$ and $x_{\mathrm{w}}$ via $\bar{y}$ as shown in the figures. For a given value of $\varkappa$, we have instability in the region connecting the two stability boundaries (i.e. above $q = -1$ and below the $\varkappa = \text{const}$ curve in Fig. 2a, and below $q = 1$ and above the $\varkappa = \text{const}$ curve in Fig. 2b). Outside this region we have stable linear behaviour. Combining the results of Figs. 2a and 2b we find that for $\varkappa > 0$ and small $x_{\mathrm{w}}^{-2}$ (wall further away from the plasma) the whole range $-1 \leqslant q \leqslant 1$ is unstable because either the $m = n = \pm 1$ or the $m = -n = \pm 1$ modes are unstable. Only if the wall is close enough to the plasma a second window of stable behaviour shows up (e.g. if $x_{\mathrm{w}}^{-2} \gtrsim 0.2$ for $\varkappa = 0.5$). Note that, although our model allows for arbitrarily large values of $\varkappa$, in a realistic tokamak the inverse aspect ratio is restricted to $\varkappa \leqslant 1$.

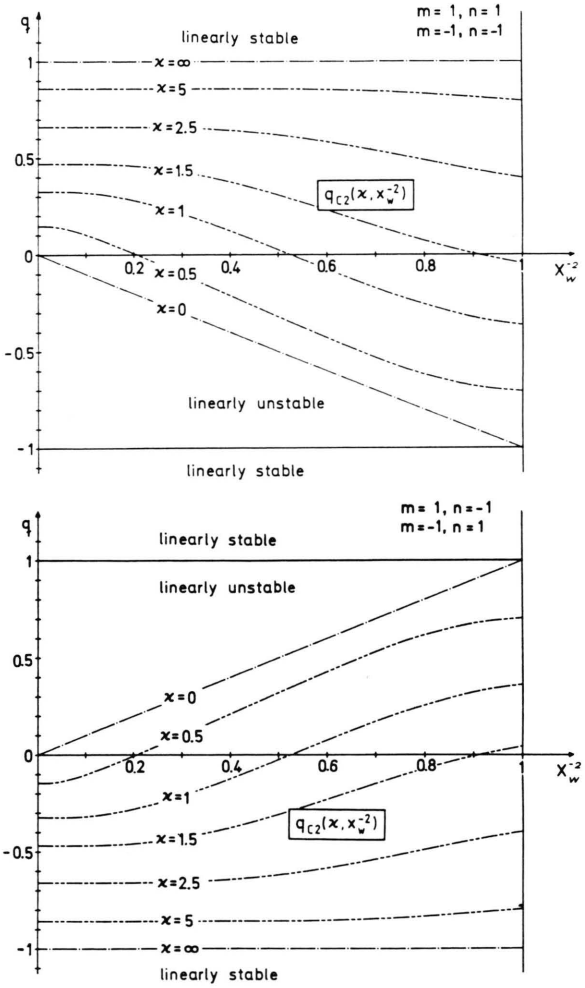


Fig. 2. a) Curves of marginal stability of the $m = n = \pm 1$ kink-modes for different values of $\varkappa$ in a $q$, $x_{\mathrm{w}}^{-2}$-plane. b) Same for the $m = -n = \pm 1$ modes.

Since the linear (and nonlinear) stability behaviour of the $m=n=-1$ and $m=-n=1$ kink-modes can be obtained by trivial transformation from that of the $m=n=1$ mode, we shall restrict our nonlinear analysis to the latter one. For this, according to (2.3c), (2.3e), (2.4), (2.5), and (1.3) the two boundaries of marginal stability are given by

$$q_{c1}=-1\,,\quad q_{c2}=\frac{2\varkappa}{\sqrt{\bar{y}^2+\varkappa^2}}-1\,, \tag{2.6}$$

where $\bar{y}$ is the solution of

$$\varkappa\bar{y}J_1'(\bar{y})+\left(\sqrt{\bar{y}^2+\varkappa^2}+\frac{2\bar{y}^2}{(1+q_{ci})G_1(\varkappa,1)\sqrt{\bar{y}^2+\varkappa^2}}\right)\cdot J_1(\bar{y})=0\,,\quad i=1,2\,. \tag{2.7}$$

According to the latter equation, $\bar{y}\to 0$ as $q\to q_{c1}$ since the last term in the brackets dominates all others then. From (2.3a) we conclude $\beta\to 1$ and $y(x)\to 0$ in this case. If the corresponding limiting solution (2.2) is inserted in the fulll linear stability equations one finds $C=0$, i.e. the limit $q=q_{c1}$ is degenerate in the sense that no marginal solution exists. However, since the existence and knowledge of the latter is the starting point for the nonlinear treatment of [1] and [2], we can apply this only at the boundary $q=q_{c2}$.

## 3. The Nonlinear Theory

For all details and definitions of the general nonlinear theory we refer to [1] and [2]. In the present paper we shall use

$$\tau=b_c-b \tag{3.1}$$

as the driving parameter, its increase from negative to positive values leading from stable to unstable.

As was shown in [2] for tokamak-like equilibria one must go up to the third order of a reductive perturbation expansion in order to obtain the normal form equation

$$\frac{d^2A}{dt^2}-\tau\gamma_T^2A=2\vartheta_T A^3 \tag{3.2}$$

for the evolution of the amplitude $A(t)$ of the mode which is the first to get unstable. The coefficients $\gamma_T$ and $\vartheta_T$ can already be obtained from the first and second order equations which will be solved now.

### 3.1 *First Order Equations and Their Solution*

For the space-dependent parts $\varphi_1(r)$ and $\mathfrak{A}_1(r)$ of the lowest order perturbations $\tilde{\varphi}_1=A_1(T)\varphi_1(r)$ and $\tilde{\mathfrak{A}}=A_1(T)\mathfrak{A}_1(r)$ we must solve the equations*

$$\begin{aligned}&\mathbb{F}_{00}(\varphi_1)=\mathbf{0}\,,\\ &\nabla\times(\nabla\times\mathfrak{A}_1)=\mathbf{0}\,,\qquad \nabla\cdot\mathfrak{A}_1=0\end{aligned} \tag{3.3a}$$

subject to the boundary conditions

$$\left.\begin{aligned}&\mathbb{P}_{10}(\varphi_1)+\boldsymbol{B}_{00}\cdot\mathbb{B}_{10}(\varphi_1)=\boldsymbol{B}_{00}^{\mathrm{v}}\cdot(\nabla\times\mathfrak{A}_1)\\ &\boldsymbol{n}_{00}\times\mathfrak{A}_1=-(\boldsymbol{n}_{00}\cdot\varphi_1)\boldsymbol{B}_{00}^{\mathrm{v}}\end{aligned}\right\}\text{ on } S_{00}\,,$$

$$\boldsymbol{n}_{00}\times\mathfrak{A}_1=\mathbf{0}\,,\qquad\text{on } C_{00}\,.$$

Herein, $S_{00}$ denotes the undisplaced plasma-vacuum boundary and $C_{00}$ the surface of the superconducting wall. In the operators $\mathbb{F}_{00}$, $\mathbb{P}_{10}$ and $\mathbb{B}_{10}$ defined in the Appendix of [2], the equilibrium quantities (1.1) must be inserted at the boundary of marginal stability setting $b=q_{c2}/\varkappa$. Since the lowest order equations are just the linear stability equations for the marginal mode, we can take their solution from Sect. 2, (2.2)–(2.3). For the $m=n=1$ mode, in the present notation we get

$$\begin{aligned}
\varphi_{1r}(r)&=\varphi_{1r}^{1,k}(u)\cosh\,,\\
\varphi_{1r}^{1,k}(u)&=C_1\{J_1'(u)+\theta_1 J_1(u)/u\}\,,\\
m_{1\varphi}(r)&=i\varphi_{1\varphi}^{1,k}(u)\sin h\,,\\
\varphi_{1\varphi}^{1,k}(u)&=iC_1\{\theta_1J_1'(u)+J_1(u)/u\}\,,\\
\varphi_{1z}(r)&=i\varphi_{1z}^{1,k}(u)\sin h\,,\\
\varphi_{1z}^{1,k}(u)&=-iC_1(\theta_1^2-1)^{1/2}J_1(u)\,,\\
\mathfrak{A}_{1z}(r)&=\mathfrak{A}_{1z}^{1,k}(\varkappa,x)\cosh\,,\\
\mathfrak{A}_{1z}^{1,k}(\varkappa,x)&=B_{0\varphi}\varphi_{1r}^{1,k}|_{S_{00}}F_1(\varkappa,x)\,,\\
\mathfrak{A}_{1\varphi}(r)&=\mathfrak{A}_{1\varphi}^{1,k}(\varkappa,x)\cosh\,,\\
\mathfrak{A}_{1\varphi}^{1,k}(\varkappa,x)&=(\varkappa x)^{-1}\mathfrak{A}_{1z}^{1,k}(\varkappa,x)+(i\psi_1^{1,k}(\varkappa,x))'\,,\\
\mathfrak{A}_{1r}(r)&=\mathfrak{A}_{1r}^{1,k}(\varkappa,x)\sin h\,,\\
\mathfrak{A}_{1r}^{1,k}(\varkappa,x)&=(\varkappa x)^{-1}i\psi_1^{1,k}(\varkappa,x)+(\mathfrak{A}_{1z}^{1,k}(\varkappa,x))'\,,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}h&:=\varphi+kz\,,\\ u&:=(\beta_0^2-1)^{1/2}\varkappa x\,,\\ \psi_1^{1,k}(\varkappa,x)&=i(1+q_c)B_{0\varphi}\varphi_{1r}^{1,k}|_{S_{00}}G_1(\varkappa,x)/\varkappa\end{aligned} \tag{3.5}$$

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* In the whole text. $\mathbb{F}_{00}$, $\mathbb{B}_{10}$ etc. are the vector operators $\mathbf{F}_{00}$, $\mathbf{B}_{10}$ etc. of Refs. [1–2], while $\mathbb{P}_{10}$ and $\mathbb{P}_1$ denote the scalar operators $\not{p}_{10}$ and $\not{p}_1$ from there.

and

$$\beta_0 = 2/(1+q_{c2}) \tag{3.6}$$

according to (2.3 b) and (2.3 c).

The constant $C_1$ which appears in the solution (3.4) is arbitrary and can be fixed by imposing the normalization condition $\int_{P2} \varphi_1, \varphi_1 \mathrm{d}\tau = 1$, the result being

$$C_1 = r_0^{-3/2}\left(\frac{\varkappa}{2\pi^2}\right)^{1/2} \cdot \left\{\int_0^1 \{(\varphi_{1r}^{1,k})^2 + (i\varphi_{1\varphi}^{1,k})^2 + (i\varphi_{1z}^{1,k})^2\} x \,\mathrm{d}x\right\}^{-1/2} . \tag{3.7}$$

### 3.2 Second Order Equations and Their Solution

According to [2], we must solve

$$\mathbb{F}_{00}(\varphi_{22}) = -\mathbb{G}_{00}(\varphi_1, \varphi_1) \tag{3.8 a}$$

in the unperturbed plasma region, and

$$\nabla \times (\nabla \times \mathfrak{A}_{22}) = \mathbf{0}, \quad \nabla \cdot \mathfrak{A}_{22} = 0 \tag{3.8 b}$$

in the unperturbed vacuum region, $\varphi_{22}(\boldsymbol{r})$ and $\mathfrak{A}_{22}(\boldsymbol{r})$ being the time-independent parts of

$$\tilde{\varphi}_{22}(\boldsymbol{r}, T) = \tfrac{1}{2} A_1^2(T)\, \varphi_{22}(\boldsymbol{r}),$$
$$\tilde{\mathfrak{A}}_{22}(\boldsymbol{r}, T) = \tfrac{1}{2} A_1^2(T)\, \mathfrak{A}_{22}(\boldsymbol{r}),$$

respectively, and $\mathbb{G}_{00}$ being defined in the Appendix of [2]. At the unperturbed plasma-vacuum boundary we must impose the boundary conditions

$$\tilde{\Pi}^l = \tilde{\Pi}^r \tag{3.9}$$

with

$$\tilde{\Pi}^2 = \{\mathbb{P}_{10}(\varphi_{22}) + \boldsymbol{B}_{00} \cdot \boldsymbol{B}_{10}(\varphi_{22}) - \overset{\vee}{\boldsymbol{B}}_{00} \cdot (\nabla \times \mathfrak{A}_{22})\} \cdot \frac{r_0^2}{B_{0\varphi}^2} \tag{3.10 a}$$

and

$$\begin{aligned}\tilde{\Pi}^r = \{&-\mathbb{P}_1(\varphi_1, \mathbb{P}_{10}(\varphi_1)) - \boldsymbol{B}_{00} \cdot \boldsymbol{B}_1(\varphi_1, \boldsymbol{B}_{10}(\varphi_1)) \\ &- (\boldsymbol{B}_{10}(\varphi_1))^2 + (\nabla \times \mathfrak{A}_1)^2 - 2\varphi_1 \cdot \nabla \{\mathbb{P}_{10}(\varphi_1) \\ &+ \boldsymbol{B}_{00} \cdot \mathbb{B}_{10}(\varphi_1) - \overset{\vee}{\boldsymbol{B}}_{00} \cdot (\nabla \times \mathfrak{A}_1)\} \\ &- \varphi_1 \varphi_1 : \nabla\nabla\, (p_{00} + B_{00}^2/2 - B^{v2}/2)\} \frac{r_0^2}{B_{0\varphi}^2},\end{aligned} \tag{3.10 b}$$

and

$$\beta^l = \beta^r \tag{3.11}$$

with

$$\beta^l = \frac{r_0}{B_{0\varphi}} [\boldsymbol{n}_{00} \times \mathfrak{A}_{22} + \boldsymbol{n}_0 \cdot \varphi_{22} \overset{\vee}{\boldsymbol{B}}_{00}] \tag{3.12 a}$$

and

$$\begin{aligned}\beta^r = \frac{-r_0}{B_{0\varphi}} \Bigg[&\left\{\frac{1}{2} \boldsymbol{n}_{00} \cdot (\varphi_1 \cdot \nabla \varphi_1) + \boldsymbol{n}_1 \cdot \varphi_1\right\} \overset{\vee}{\boldsymbol{B}}_{00} \\ &+ \boldsymbol{n}_{00} \cdot \varphi_1 (\varphi_1 \cdot \nabla \overset{\vee}{\boldsymbol{B}}_{00}) + \{\boldsymbol{n}_1 \times \mathfrak{A}_1 \\ &+ \boldsymbol{n}_{00} (\varphi_1 \cdot \nabla) \mathfrak{A}_1 + \boldsymbol{n}_{00} \cdot \varphi_1 (\nabla \times \mathfrak{A}_1)\}\Bigg].\end{aligned} \tag{3.12 b}$$

At the surface of the superconducting wall (127) of [2] or

$$\boldsymbol{n}_{00} \times \mathfrak{A}_{22} = \mathbf{0} \tag{3.13}$$

must be satiesfied. The normal vector $\boldsymbol{n}_1$ appearing in (5.9) is defined by (70) of ref. [2].

Our further procedure will be as follows: We shall first evaluate all terms which are determined by the first order quantities $\varphi_1$ and $\mathfrak{A}_1$ and are thus playing the role of inhomogeneities, i.e. $\mathbb{G}_{00}(\varphi_1, \varphi_1)$ and the right hand sides $\tilde{\Pi}^r$ and $\beta^r$ of the boundary conditions. We shall see that these terms are either independent both on $\varphi$ and $z$ or depend on them in proportionality to $\cos(2h) = \cos[2(\varphi + kz)]$. This implies the ansatz

$$\begin{aligned}\mathfrak{A}_{22}(\boldsymbol{r}) &= \mathfrak{A}_{22}^{0,0}(x) + \mathfrak{A}_{22}^{2,2k}(x) \exp(i2h), \\ \varphi_{22}(\boldsymbol{r}) &= \varphi_{22}^{0,0}(x) + \varphi_{22}^{2,2k}(x) \exp(i2h)\end{aligned} \tag{3.14}$$

for $\varphi_{22}$ and $\mathfrak{A}_{22}$. (We can use complex notation since the equations to be solved are linear. Note that for evaluating the nonlinear coefficient one must use only the real parts.)

Before entering the details of the procedure just described we collect the evaluations of certain terms which will be needed repeatedly. Using some definitions given in the Appendix of [2], and (3.4) – (3.5) (remember $\operatorname{div} \varphi_1 = 0$), we find

$$\mathbb{B}_{10}(\varphi_1) = \frac{B_{0\varphi}}{r_0}(1 + q_c)\, i\varphi_1,$$

$$\mathbb{P}_{10}(\varphi_1) = 2\frac{B_{0\varphi}^2}{r_0} x \varphi_{1r}^{1k} \cos h,$$

$$\mathbb{B}_1(\varphi_1, \mathbb{B}_{10}(\varphi_1)) = -\frac{B_{0\varphi}}{r_0^2}(1 + q_c)\{(\varphi_{1r}^{1k}\, i\varphi_{1\varphi}^{1k})^{\bullet} \boldsymbol{e}_\varphi + x^{-1}(x\varphi_{1r}^{1k}\, i\varphi_{1z}^{1k})^{\bullet} \boldsymbol{e}_z\},$$

$$\mathbb{P}_1(\varphi_1, \mathbb{P}_{10}(\varphi_1)) = 2\frac{B_{0\varphi}^2}{r_0^2}(-\varphi_{1r}^{1k})(x\varphi_{1r}^{1k})^{\bullet},$$

$$\begin{aligned}\varphi_1 \cdot \nabla \varphi_1 = (2xr_0)^{-1}\{([(x\varphi_{1r}^{1k}\varphi_{1r}^{1k})^{\bullet} - (i\varphi_{1\varphi}^{1k})^2]\\ - [(\varphi_{1r}^{1k})^2 - (i\varphi_{1\varphi}^{1k})^2]\cos 2h)\boldsymbol{e}_r\\ + x[\varphi_{1r}^{1k}(i\varphi_{1\varphi}^{1k})^{\bullet} - (\varphi_{1r}^{1k})^{\bullet} i\varphi_{1\varphi}^{1k}]\sin 2h\,\boldsymbol{e}_\varphi\\ + [x\varphi_{1r}^{1k}(i\varphi_{1z}^{1k})^{\bullet} - (x\varphi_{1r}^{1k})^{\bullet} i\varphi_{1r}^{1k}]\sin 2h\,\boldsymbol{e}_z\},\end{aligned}$$

$$[\nabla x \boldsymbol{B}_{00}(\varphi_1)]x\mathbb{B}_{10}(\varphi_1) = 0\,. \tag{3.15}$$

The dot stands for the derivative with respect to $x$, and

$$\begin{aligned} i\varphi_1 = -\varphi_{1r}^{1,k}\sinh h\,\boldsymbol{e}_r + i\varphi_{1\varphi}^{1,k}\cosh h\,\boldsymbol{e}_\varphi\\ + i\varphi_{1z}^{1,k}\cosh h\,\boldsymbol{e}_z \end{aligned}\tag{3.16}$$

according to (3.4).

Using these relations, after some calculation one finds

$$\begin{aligned}\mathbb{G}_{00}(\varphi_1,\varphi_1) = \frac{B_{0\varphi}^2}{r_0^3}\{2[\varphi_{1r}^{1k}(x\varphi_{1r}^{1k})^{\bullet}]^{\bullet}\\ + (1+q_c)[3(\varphi_{1r}^{1k} i\varphi_{1\varphi}^{1k})^{\bullet} + x(\varphi_{1r}^{1k} i\varphi_{1\varphi}^{1k})^{\bullet\bullet}\\ + b_c\{x^{-1}(x\varphi_{1r}^{1k} i\varphi_{1z}^{1k})^{\bullet}\}^{\bullet}]\}\boldsymbol{e}_r,\end{aligned}\tag{3.17}$$

$$\tilde{\Pi}^r = \tilde{\Pi}_0^r + \Pi_2^r\cos 2h \tag{3.18}$$

with

$$\begin{aligned}\tilde{\Pi}_0^r = -(2+Q_c^2\tilde{G}_1(\varkappa,1))\varphi_{1r}^{1,k}(\varphi_{1r}^{1,k})^{\bullet} - 2Q_c\varphi_{1r}^{1,k} i\varphi_{1\varphi}^{1,k}\\ - Q_c^2[(i\varphi_{1\varphi}^{1,k})^2 + (i\varphi_{1z}^{1,k})^2]/2 + Q_c\{2\tilde{G}_1(\varkappa,1)\\ + Q_c[(1+\varkappa^2)(\tilde{G}_1(\varkappa,1))^2/2 - (1+\tilde{G}_1(\varkappa,1))]\}\\ \cdot(\varphi_{1r}^{1k})^2,\end{aligned}\tag{3.19a}$$

$$\begin{aligned}\tilde{\Pi}_2^r = Q_c[b_c\varphi_{1r}^{1,k} i\varphi_{1z}^{1,k} - \varphi_{1r}^{1,k}(i\varphi_{1\varphi}^{1,k} + b_c i\varphi_{1z}^{1,k})^{\bullet}\\ + (\varphi_{1r}^{1,k})^{\bullet}(i\varphi_{1\varphi}^{1,k} + b_c i\varphi_{1z}^{1,k})]\\ + Q_c^2[\tilde{G}_1(\varkappa,1)\varphi_{1r}^{1,k}(\varphi_{1r}^{1,k})^{\bullet}\\ - \{(i\varphi_{1\varphi}^{1,k})^2 + (i\varphi_{1z}^{1,k})^2\}/2]\\ + [Q_c^2\{(1+\varkappa^2)(\tilde{G}_1(\varkappa,1))^2/2 + (\tilde{G}_1(\varkappa,1)-\tfrac{3}{2})\}\\ + 2(1+Q_c\tilde{G}_1(\varkappa,1))](\varphi_{1r}^{1,k})^2,\end{aligned}\tag{3.19b}$$

and

$$\beta^r = \beta_0^r + \beta_2^r\cos 2h \tag{3.20}$$

with

$$\begin{aligned}\beta_0^r = \{[\tfrac{5}{4} + Q_c\tilde{G}_1(\varkappa,1)](\varphi_{1r}^{1,k})^2 + \tfrac{1}{2}\varphi_{1r}^{1,k}(\varphi_{1r}^{1,k})^{\bullet}\\ - \tfrac{1}{4}(i\varphi_{1\varphi}^{1,k})^2\}\boldsymbol{e}_\varphi + \{[b_c/4 + Q_c G_1(\varkappa,1)](\varphi_{1r}^{1,k})^2\\ + (b_c/2)\varphi_{1r}^{1,k}(\varphi_{1r}^{1,k})^{\bullet} + [F_1'(\varkappa,1) - Q_c\varkappa^{-1}\tilde{G}_1(\varkappa,1)]\\ \cdot\varphi_{1r}^{1,k} i\varphi_{1\varphi}^{1,k} - (b_c/4)(i\varphi_{1\varphi}^{1,k})^2\}\boldsymbol{e}_z\end{aligned}\tag{3.21a}$$

and

$$\begin{aligned}\beta_2^r = \{[\varkappa F_1'(\varkappa,1) - \tfrac{1}{4}](\varphi_{1r}^{1,k})^2 - \varphi_{1r}^{1,k}(\varphi_{1r}^{1,k})^{\bullet}\\ + \tfrac{1}{4}(i\varphi_{1\varphi}^{1,k})^2\}\boldsymbol{e}_\varphi + \{[Q_c(1+\varkappa^2)\varkappa^{-1}\tilde{G}_1(\varkappa,1)\\ - F_1'(\varkappa,1) - \tfrac{5}{4}b_c](\varphi_{1r}^{1,k})^2 - b_c\varphi_{1r}^{1,k}(\varphi_{1,r}^{1,k})^{\bullet}\\ - [F_1'(\varkappa,1) - Q_c\varkappa^{-1}\tilde{G}_1(\varkappa,1)]\varphi_{1r}^{1,k} i\varphi_{1\varphi}^{1,k}\\ + (b_c/4)(i\varphi_{1\varphi}^{1,k})^2\}\boldsymbol{e}_z\,.\end{aligned}\tag{3.21b}$$

*Solution of the Plasma Equation*

With our ansatz (3.14) and the result (3.17), (3.8a) can be decomposed into

$$\mathbb{F}_{00}(\varphi_{22}^{0,0}) = -\mathbb{G}_{00}(\varphi_1,\varphi_1)\,, \tag{3.22a}$$

$$\mathbb{F}_{00}(\varphi_{22}^{2,2k}\exp[i2h]) = \boldsymbol{0}\,. \tag{3.22b}$$

Equation (3.22b) is just the marginal mode equation, i.e. $\varphi_{22}^{2,2k}\cdot\exp\{i2h\}$ must be a marginal mode with $m = n = 2$. It can be obtained from our solution (3.4) by the transition $\varphi\to 2\varphi$, $k\to 2k$ and $\varkappa\to 2\varkappa$. Since the argument of the Bessel functions thus obtained changes from real to imaginary values as $q_c$ changes from $q_c<0$ to $q_c>0$, one must treat these two cases separately. We get

$$\begin{aligned}\Re\mathrm{e}[\varphi_{22}^{2,2k}\exp(i2h)] = \varphi_{22r}^{2,2k}\cos 2h\,\boldsymbol{e}_r + i\varphi_{22\varphi}^{2,2k}\sin 2h\,\boldsymbol{e}_\varphi\\ + i\varphi_{22z}^{2,2k}\sin 2h\,\boldsymbol{e}_z\end{aligned}\tag{3.23}$$

with

$$\begin{aligned}\varphi_{22r}^{2,2k} &= C_2\{J_2'(w) + 2\beta_0 J_2(w)/w\},\\ \varphi_{22\varphi}^{2,2k} &= iC_2\{\beta_0 J_2'(w) + 2J_2(w)/w\},\\ \varphi_{22z}^{2,2k} &= -iC_2\{\beta_0^2 - 1\}^{1/2}J_2(w),\\ w &:= (\beta_0^2-1)^{1/2}2\varkappa x,\quad \beta_0 = (1+q_{c2})^{-1}\end{aligned}\tag{3.24a}$$

for $q_c \leqslant 0$ and

$$\begin{aligned}\varphi_{22r}^{2,2k} &= C_2\{I_2'(v) + 2\beta_0 I_2(v)/v\},\\ \varphi_{22\varphi}^{2,2k} &= iC_2\{\beta_0 I_2'(v) + 2I_2(v)/v\},\\ \varphi_{22z}^{2,2k} &= iC_2(1-\beta_0^2)^{1/2}I_2(v),\\ v &:= (1-\beta_0^2)^{1/2}2\varkappa x\end{aligned}\tag{3.24b}$$

for $q_c \geqslant 0$. The constant $C_2$ will be fixed when we evaluate the boundary conditions.

We now come to solving (3.22a). Using

$$\mathbb{B}_{10}(\varphi_{22}^{0,0}) = -\frac{1}{r_0}D(x)\boldsymbol{B}_{00}\,,$$

$$j_{00}\times\mathbb{B}_{10}(\varphi_{22}^{0,0}) = 2\frac{B_{0\varphi}^2}{r_0^2}xD(x)\boldsymbol{e}_r\,,$$

$$\begin{aligned}(\nabla\times\mathbb{B}_{10}(\varphi_{22}^{0,0}))\times\boldsymbol{B}_{00} = \frac{B_{0\varphi}^2}{r_0^2}\{(x^2+b_c^2)D^{\bullet}(x)\\ + 2xD(x)\}\boldsymbol{e}_r\,,\end{aligned}$$

$$\nabla \mathbb{P}_{10}(\varphi_{22}^{0,0}) = \frac{B_{0\varphi}^2}{3 r_0^2}\{-16 x D(x) + 5(1-x^2) D^\bullet(x)\} \boldsymbol{e}_r ,$$

where

$$D(x) = x^{-1}(x\varphi_{22r}^{0,0})^\bullet , \tag{3.25}$$

we find

$$\mathbb{F}_{00}(\varphi_{22}^{0,0}) = -\frac{2}{3}\frac{B_{0\varphi}^2}{r_0^2}\{(x^2-\alpha^2)D(x)\}^\bullet \boldsymbol{e}_r , \tag{3.26}$$

$$\alpha^2 := (5+3b_c^2)/2 \geqslant 5/2 .$$

Inserting this and (3.17) in (3.8 a), we have to solve the equation

$$\{(x^2-\alpha^2)D(x)\}^\bullet = \frac{3}{2r_0}\{2[\varphi_{1r}^{1k}(x\varphi_{1r}^{1k})^\bullet]^\bullet + (1+q_c)[3(\varphi_{1r}^{1k} i\varphi_{1\varphi}^{1k})^\bullet + x(\varphi_{1r}^{1k} i\varphi_{1\varphi}^{1k})^{\bullet\bullet} + b_c\{x^{-1}(x\varphi_{1r}^{1k} i\varphi_{1z}^{1k})^\bullet\}^\bullet]\} ,$$

which has the solution

$$D(x) = \left\{\frac{2}{r_0}C_{01} + \frac{3}{2r_0}\{2\varphi_{1r}^{1k}(x\varphi_{1r}^{1k})^\bullet + (1+q_c) \cdot [2\varphi_{1r}^{1k} i\varphi_{1\varphi}^{1k} + x(\varphi_{1r}^{1k} i\varphi_{1\varphi}^{1k})^\bullet \cdot b_c x^{-1}(x\varphi_{1r}^{1k} i\varphi_{1z}^{1k})^\bullet]\}\right\}\{x^2-\alpha^2\}^{-1} . \tag{3.27}$$

Integrating (3.25), we finally obtain

$$\varphi_{22r}^{0,0}(x) = \frac{C_{01}}{r_0}x^{-1}\ln\left(1-\frac{x^2}{\alpha^2}\right) + x^{-1}\int_0^x \frac{\mathrm{d}\Gamma(t)/\mathrm{d}t}{t^2-\alpha^2}\,\mathrm{d}t \tag{3.28}$$

with

$$\Gamma(x) := \frac{3}{2r_0}\{(x\varphi_{1r}^{1k})^2 + (1+q_c)[(x^2\varphi_{1r}^{1k} i\varphi_{1\varphi}^{1k}) + b_c(x\varphi_{1r}^{1k} i\varphi_{1z}^{1k})]\} , \tag{3.29}$$

where an integration constant has been chosen such that the solution has no singularity within the plasma. The constant $C_{01}$ will be fixed through the boundary conditions.

Note that the components $\varphi_{22,\varphi}^{0,0}$ and $\varphi_{22,z}^{0,0}$ remain undetermined at this stage. This does not cause any difficulties since they do not appear anywhere in the further procedure.

*Solution of the Vacuum Equations*

Using our ansatz (3.14), equations (3.8 b) yield

$$0 = -\frac{1}{x}(x\mathfrak{A}_{22z}^{2,2k\bullet})^\bullet + \frac{4}{x^2}\mathfrak{A}_{22z}^{2,2k} + 4\varkappa^2\mathfrak{A}_{22z}^{2,2k} ,$$

$$0 = \left(\frac{4}{x^2}+4\varkappa^2\right)\mathfrak{A}_{22\varphi}^{2,2k} - \frac{1}{x}(x\mathfrak{A}_{22\varphi}^{2,2k\bullet})^\bullet + \frac{1}{x^2}\mathfrak{A}_{22\varphi}^{2,2k\bullet} - \frac{4i}{x^2}\mathfrak{A}_{22r}^{2,2k} ,$$

$$0 = \left(\frac{4}{x^2}+4\varkappa^2\right)\mathfrak{A}_{22r}^{2,2k} - \frac{1}{x}(x\mathfrak{A}_{22r}^{2,2k\bullet})^\bullet - \frac{1}{x^2}\mathfrak{A}_{22r}^{2,2k\bullet} + \frac{4i}{x^2}\mathfrak{A}_{22\varphi}^{2,2k}$$

for $\boldsymbol{\mathfrak{A}}_{22}^{2,2k}$ and

$$0 = \left\{\frac{1}{x}[x\mathfrak{A}_{22\varphi}^{0,0}]^\bullet\right\}^\bullet \boldsymbol{e}_\varphi - \frac{1}{x}\{x\mathfrak{A}_{22z}^{0,0\bullet}\}^\bullet \boldsymbol{e}_z ,$$

$$0 = \{x\mathfrak{A}_{22r}^{0,0}\}^\bullet$$

for $\boldsymbol{\mathfrak{A}}_{22}^{0,0}$. Solutions which satisfy the boundary condition (3.13) are given by

$$\begin{aligned}\mathfrak{A}_{22z}(\boldsymbol{r}) &= \mathfrak{A}_{22z}^{2,2k}(2\varkappa,x)\cos(2h) ,\\ \mathfrak{A}_{22\varphi}(\boldsymbol{r}) &= \mathfrak{A}_{22\varphi}^{2,2k}(2\varkappa,x)\cos(2h) ,\\ \mathfrak{A}_{22r}(\boldsymbol{r}) &= \mathfrak{A}_{22r}^{2,2k}(2\varkappa,x)\sin(2h)\end{aligned} \tag{3.30}$$

with

$$\mathfrak{A}_{22z}^{2,2k}(2\varkappa,x) = \frac{C_{2z}B_{0\varphi}}{r_0}F_2(2\varkappa,x) ,$$

$$\mathfrak{A}_{22\varphi}^{2,2k}(2\varkappa,x) = (\varkappa x)^{-1}\mathfrak{A}_{22z}^{2,2k}(2\varkappa,x) + (i\psi_2^{2,2k}(2\varkappa,x))' ,$$

$$\mathfrak{A}_{22r}^{2,2k}(2\varkappa,x) = (\varkappa x)^{-1} i\psi_2^{2,2k}(2\varkappa,x) + (\mathfrak{A}_{22z}^{2,2k}(2\varkappa,x))' ,$$

$$\psi_2^{2,2k}(2\varkappa,x) = \frac{iC_\psi B_{0\varphi}}{r_0}\tilde{G}_2(2\varkappa,x) \tag{3.31 a}$$

and

$$\mathfrak{A}_{22r}^{0,0}(x) = \frac{C_r B_{0\varphi}}{r_0}x^{-1} ,$$

$$\mathfrak{A}_{22\varphi}^{0,0}(x) = \frac{C_\varphi B_{0\varphi}}{r_0}(x - x_w^2 x^{-1}) , \tag{3.31 b}$$

$$\mathfrak{A}_{22z}^{0,0}(x) = \frac{C_z B_{0\varphi}}{r_0}\ln(x x_w^{-1}) .$$

*Evaluation of the Boundary Conditions*

Using (3.23) and (3.30) we first determine the left hand sides $\tilde{\Pi}^l$ and $\boldsymbol{\beta}^l$ of the boundary conditions, (3.9) and (3.10). All quantities have to be evaluated at $x = 1$, and we obtain

$$\tilde{\Pi}^l = \tilde{\Pi}_0^l + \tilde{\Pi}_0^l \cos(2h), \quad \beta^l = \beta_0^l + \beta_2^l \cos(2h) \tag{3.32}$$

with

$$\tilde{\Pi}_0^l = r_0 \left\{ 2\varphi_{22r}^{0,0} + \frac{1}{B_{0\varphi}} (C_z - 2b_c C_\varphi) - (1+b_c^2) D(1) \right\},$$

$$\tilde{\Pi}_2^l = r_0 \left\{ 2\varphi_{22r}^{2,2k} [1 + Q_c (\varphi_{22r}^{2,2k})^{-1} (i\varphi_{22\varphi}^{2,2k} + b_c i\varphi_{22z}^{2,2k})] - \frac{2}{B_{0\varphi}} Q_c i \Psi_2^{2,2k} \right\} \tag{3.33a}$$

and

$$\beta_0^l = r_0 \left\{ \varphi_{22r}^{0,0} - \frac{1}{B_{0\varphi}} \mathfrak{A}_{22z}^{0,0} \right\} \boldsymbol{e}_\varphi + r_0 \left\{ b_c \varphi_{22r}^{0,0} + \frac{1}{B_{0\varphi}} \mathfrak{A}_{22\varphi}^{0,0} \right\} \boldsymbol{e}_z,$$

$$\beta_2^l = r_0 \left\{ \varphi_{22r}^{2,2k} - \frac{1}{B_{0\varphi}} \mathfrak{A}_{22z}^{2,2k} \right\} \boldsymbol{e}_\varphi + r_0 \left\{ b_c \varphi_{22r}^{2,2k} + \frac{1}{B_{0\varphi}} \mathfrak{A}_{22\varphi}^{2,2k} \right\} \boldsymbol{e}_z. \tag{3.33b}$$

Defining

$$\hat{\tilde{\Pi}}_0^r = \tilde{\Pi}_0^r - 2 \int_0^1 \frac{r_0 \Gamma^\bullet(x)}{x^2 - \alpha^2} \mathrm{d}x + \frac{1 + b_c^2}{1 - \alpha^2} \Gamma^\bullet(x)|_{x=1} \tag{3.34}$$

and

$$\hat{\beta}_{0\varphi}^r = \beta_{0\varphi}^r - \int_0^1 \frac{r_0 \Gamma^\bullet(x)}{x^2 - \alpha^2} \mathrm{d}x, \quad \hat{\beta}_{0z}^r = \beta_{0z}^r - b_c \int_0^1 \frac{r_0 \Gamma^\bullet(x)}{x^2 - \alpha^2} \mathrm{d}x, \tag{3.35}$$

the angle-independent contributions to (3.9) and (3.11) lead to the system of equations

$$\begin{bmatrix} -2b_c & 1 & 2\{\ln(1-\alpha^{-2}) - \frac{2}{3}\} \\ 1 - x_w^2 & 0 & b_c \ln(1-\alpha^{-2}) \\ 0 & \ln x_w & \ln(1-\alpha^{-2}) \end{bmatrix} \cdot \begin{bmatrix} C_\varphi \\ C_z \\ C_{01} \end{bmatrix} = \begin{bmatrix} \hat{\tilde{\Pi}}_0^r \\ \hat{\beta}_{0z}^r \\ \hat{\beta}_{0\varphi}^r \end{bmatrix},$$

which has the solution

$$C_{01} = \left\{ \hat{\tilde{\Pi}}_0^r + \frac{2b_c}{1 - x_w^2} \hat{\beta}_{0z}^r - \frac{1}{\ln x_w} \hat{\beta}_{0\varphi}^r \right\} \cdot \left\{ 2[\ln(1-\alpha^{-2}) - \tfrac{2}{3}] + \left[ \frac{2b_c}{1 - x_w^2} - \frac{1}{\ln x_w} \right] \ln(1-\alpha^{-2}) \right\}^{-1},$$

$$C_z = \frac{1}{\ln x_w} \{\hat{\beta}_{0\varphi}^r - \ln(1-\alpha^{-2}) C_{01}\},$$

$$C_\varphi = \frac{1}{1 - x_w^2} \{\hat{\beta}_{0z}^r - b_c \ln(1-\alpha^{-2}) C_{01}\}. \tag{3.36}$$

Similarly, for the terms $\sim \cos(2h)$ one finds

$$\begin{bmatrix} -(2\varkappa)^{-1} & \varkappa^{-1} b_c (\varphi_{22r}^{2,2k}/C_2) \\ 2Q_c \tilde{G}_2(2\varkappa, 1) & 0 \quad \hat{\alpha} 2(\varphi_{22r}^{2,2k}/C_2) \\ 0 & -1 \quad (\varphi_{22r}^{2,2k}/C_2) \end{bmatrix} \cdot \begin{bmatrix} C_\psi \\ C_{2z} \\ r_0 C_2 \end{bmatrix} = \begin{bmatrix} \beta_{2z}^r \\ \tilde{\Pi}_2^r \\ \beta_{2\varphi}^r \end{bmatrix},$$

where

$$\hat{\alpha} = 1 + Q_c (\varphi_{22\varphi}^{2,2k})^{-1} (i\varphi_{22\varphi}^{2,2k} + b_c i\varphi_{22z}^{2,2k}).$$

The solution of the latter system is

$$C_2 = r_0^{-1} \frac{4Q_c \tilde{G}_2(2\varkappa, 1)[\varkappa \beta_{2z}^r + \beta_{2\varphi}^r] + \tilde{\Pi}_2^r}{4Q_c^2 \tilde{G}_2(2\varkappa, 1) + 2\hat{\alpha}} \cdot \{\varphi_{22r}^{2,2k}/C_2\}^{-1},$$

$$C_{2z} = \frac{\tilde{\Pi}_2^r + 4\varkappa Q_c \tilde{G}_2(2\varkappa, 1)\beta_{2z}^r}{4Q_c^2 \tilde{G}_2(2\varkappa, 1) + 2\hat{\alpha}} \frac{(4q_c Q_c \tilde{G}_2(2\varkappa, 1) + 2\hat{\alpha})\beta_{2\varphi}^r}{4Q_c^2 \tilde{G}_2(2\varkappa, 1) + 2\hat{\alpha}},$$

$$C_\psi = \frac{2[Q_c \tilde{\Pi}_2^r - 2\hat{\alpha}(\varkappa \beta_{2z}^r + \beta_{2\varphi}^r)]}{4Q_c^2 \tilde{G}_2(2\varkappa, 1) + 2\hat{\alpha}}. \tag{3.37}$$

Inserting the results (3.36) and (3.37) in (3.24), (3.28) and (3.31) completes our solution of the second order equations.

### 3.3 Determination of the Coefficients $\gamma_T^2$ and $\vartheta_T$

For determining the linear coefficient $\gamma_T^2$, in principle the general formula (128) of [2] could be used. However, in our case the situation is appreciably simplified by the fact that we have the full linear dispersion relation (2.5) at hand. Although this one was derived under the special assumption of incompressibility, we can nevertheless make use of it since (128) of [2] contains only the marginal mode for which all terms involving compressibility vanish.

It is rather straight forward to derive from it the result

$$\gamma_T^2 = \frac{\varkappa B_{0\varphi}^2}{\varrho_0 r^2} (1 + q_{c2}). \tag{3.38}$$

The nonlinear coefficient $\vartheta_T$ in (3.2) is given by (112) of [2] and can be decomposed into

$$\vartheta_T = \frac{B_{0\varphi}^2}{\varrho_0 r_0^7} \sum_{n=1}^{32} C_n^{(\vartheta)} . \tag{3.39}$$

The calculation of the 32 coefficients is very lengthy and is omitted here. Interested readers are referred to [3]. We only remark that the sum in (3.39) depends on the parameters $\varkappa$, $x_w$ and $b_c$. For the following discussion it was evaluated numerically.

## 4. Discussion of Nonlinear Results

To the lowest significant order, the nonlinear behaviour of linearly unstable modes near the boundary of marginal stability is determined by the amplitude equation (3.2). Since we consider the linearly unstable regime ($\tau \gamma_T^2 > 0$), the qualitative nonlinear behaviour is completely determined by the sign of $\vartheta_T$. It was shown in [1] that we have nonlinear oscillations for $\vartheta_T < 0$ and explosive instability ($A \to \infty$ after a finite lapse of time) for $\vartheta_T > 0$. Bifurcating equilibria are obtained from (3.2) for $d^2A/dt^2 = 0$ or

$$\tau = -\left(\frac{2\,\vartheta_T}{\gamma_T^2}\right) A^2 . \tag{4.1}$$

We see that $\vartheta_T < 0$ or $\vartheta_T > 0$ lead to bifurcation into the linearly stable ($\tau < 0$) or unstable ($\tau > 0$) regime respectively. If $\vartheta_T$ is given, the curvature of the parabola (4.1) is inversely proportional to $\gamma_T^2$

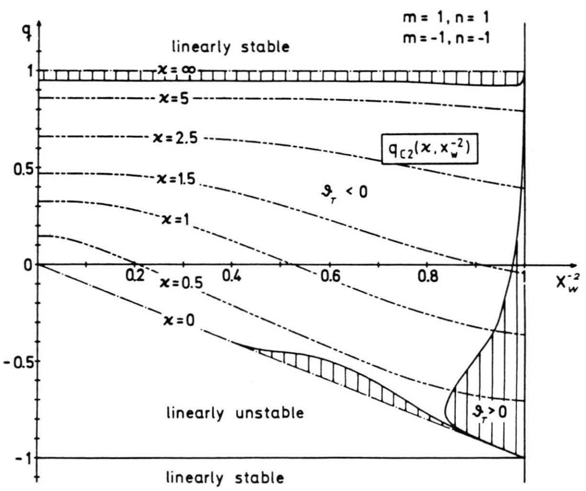


Fig. 3. Sign of $\vartheta_T$ inserted in the stability diagram of Figure 2a. Hatched areas: $\vartheta_T > 0$. Unhatched areas: $\vartheta_T < 0$.

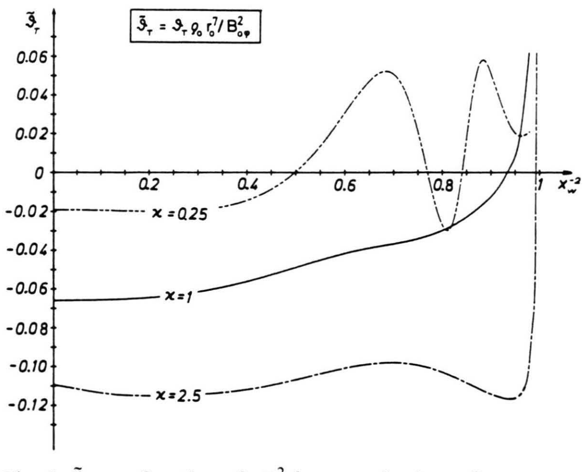


Fig. 4. $\tilde{\vartheta}_T$ as a function of $x_w^{-2}$ for several values of $\varkappa$.

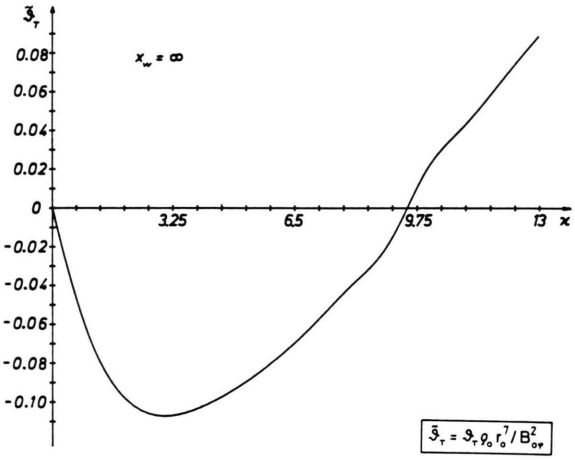


Fig. 5. $\tilde{\vartheta}_T$ as a function of $\varkappa$ for $x_w = \infty$.

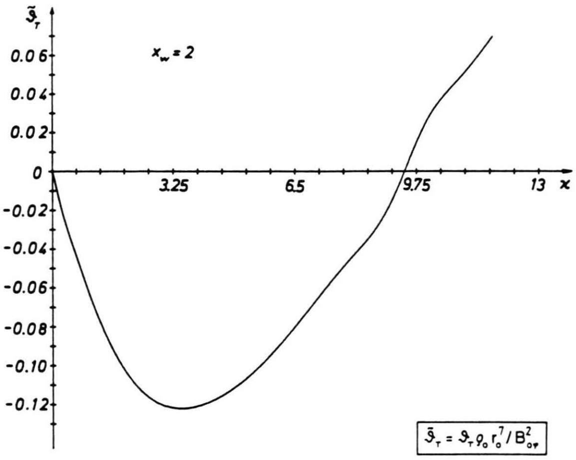


Fig. 6. $\tilde{\vartheta}_T$ as a function of $\varkappa$ for $x_w = 2$.

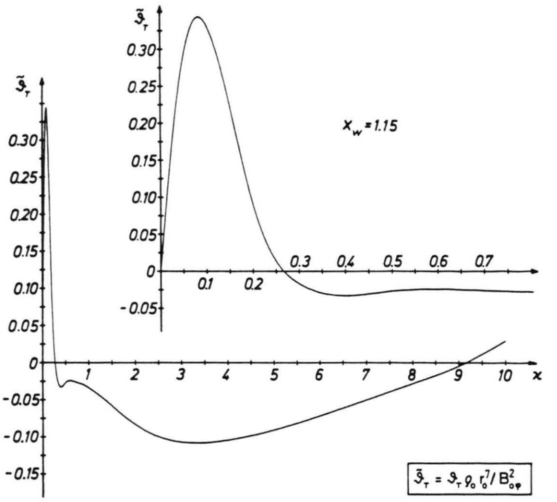


Fig. 7. $\tilde{\vartheta}_{\mathrm{T}}$ as a function of $\varkappa$ for $x_{\mathrm{w}} = 1.15$, enlarged region around $\varkappa = 0$ shown as an insert.

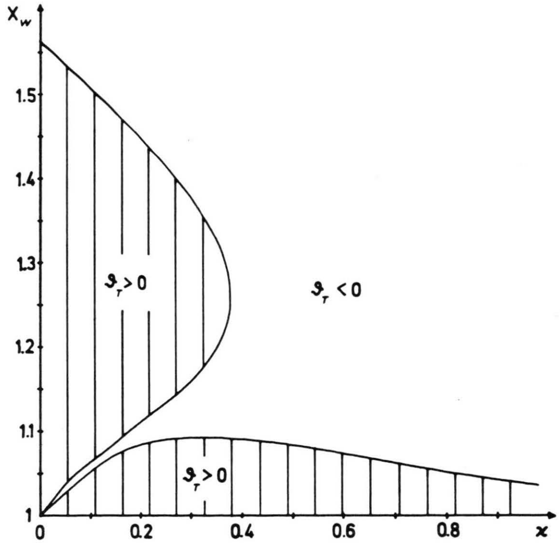


Fig. 9. Sign of $\vartheta_{\mathrm{T}}$ in a $\varkappa$, $x_{\mathrm{w}}$-plane.

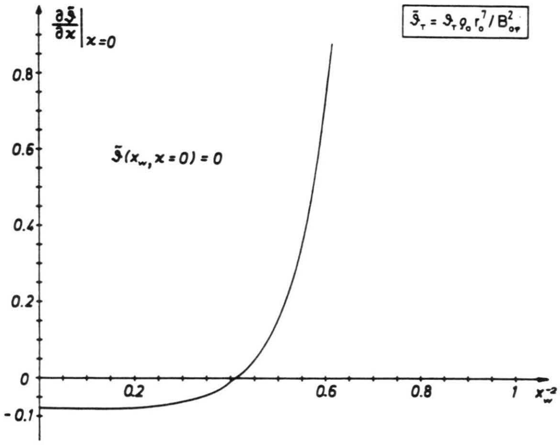


Fig. 8. $\mathrm{d}\tilde{\vartheta}_{\mathrm{T}}/\mathrm{d}\varkappa$ for $\varkappa = 0$ as a function of $x_{\mathrm{w}}^{-2}$.

which means that for small growth rates the distance of the bifurcating branch from the original one ($A = 0$) becomes large.

We now concentrate our discussion on the sign and magnitude of $\vartheta_{\mathrm{T}}$. For each point of the different marginal boundaries $\varkappa = \mathrm{const}$ in the linear stability diagram Fig. 2a, $\vartheta_{\mathrm{T}}$ was evaluated numerically. Figure 3 shows where $\vartheta_{\mathrm{T}} > 0$ (hatched areas) and $\vartheta_{\mathrm{T}} < 0$ (unhatched areas). As for the linearly unstable regimes below the curve $\varkappa = 0$ and above the curve $\varkappa = \infty$ we cannot make any nonlinear predictions since there are no marginal states.

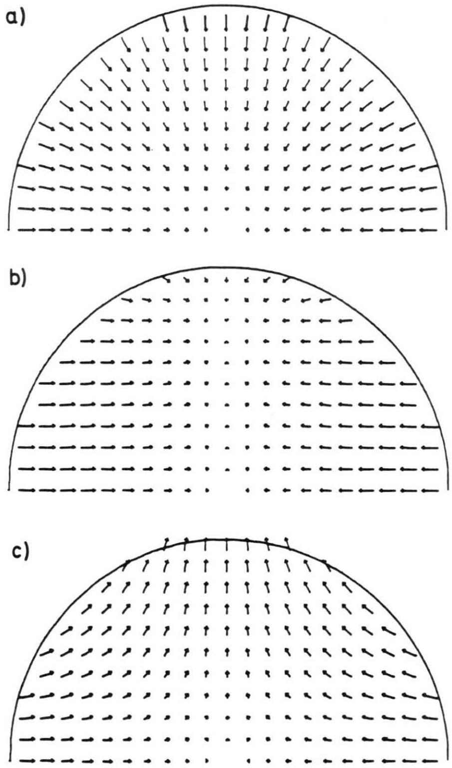


Fig. 10. Projection of $\varepsilon^2 \xi_2$ on the plane $z = 0$ a) for $x_{\mathrm{w}} = \infty$, $\varkappa = 1$; b) for $x_{\mathrm{w}} = 2$, $\varkappa = 0.5$; c) for $x_{\mathrm{w}} = 1.25$, $\varkappa = 0.2$.

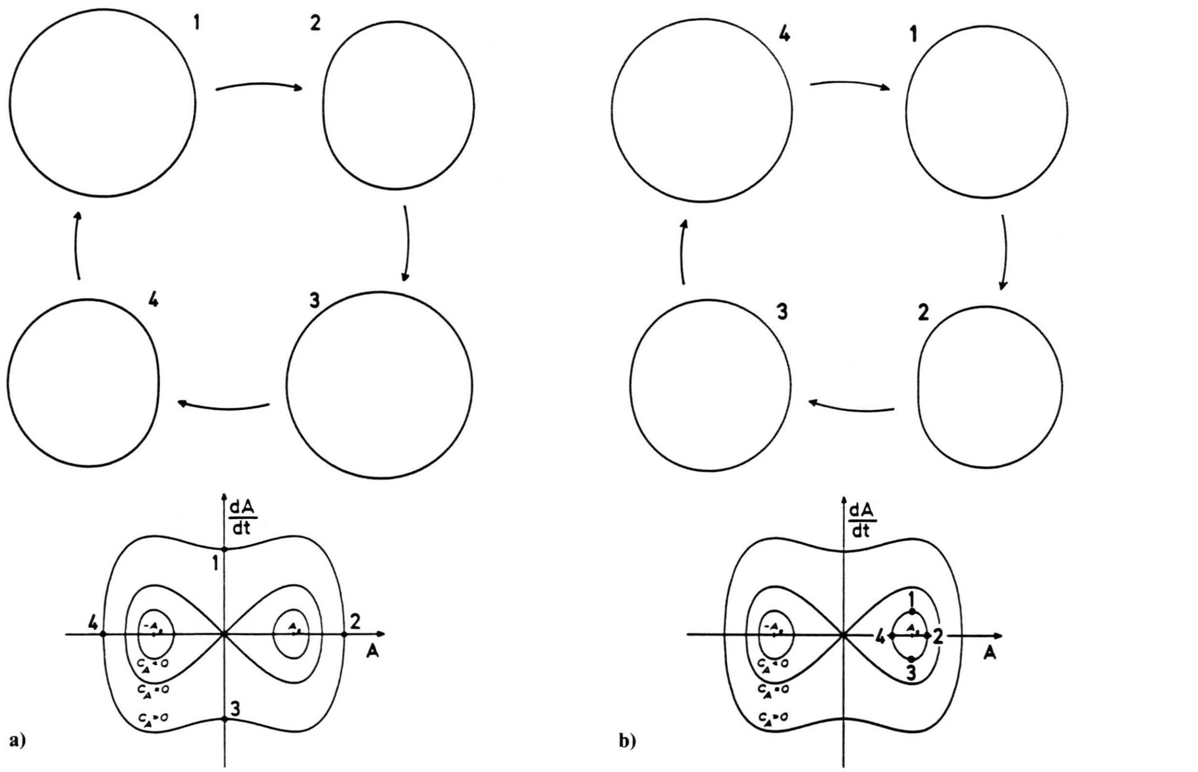


Fig. 11. Changes of the shape of the plasma boundary and corresponding phase space diagram $dA/dt$ versus $A$ during a full nonlinear oscillation in the regime $\vartheta_T < 0$ of nonlinear stabilization. a) The phase space trajectory encircles both bifurcating equilibria. $A = A_0$ and $A = -A_0$. b) Only the equilibrium $A = A_0$ is encircled.

Let us consider the curve $\varkappa = 0.5$ as a special example. The $m = n = 1$ mode becomes linearly unstable if we go slightly below this curve, but it is nonlinearly stabilized if the superconducting wall is far enough from the plasma ($x_w^{-2} \lesssim 0.85$). For $x_w^{-2} \lesssim 0.2$ we cannot say whether or not this nonlinear stabilization is sure to help because according to Fig. 2b the curve $\varkappa = 0.5$ enters the linearly unstable regime of the $m = -n = \pm 1$ mode there. For $\varkappa = 0.5$ and $x_w^{-2} \gtrsim 0.85$, the nonlinear stabilization turns over into destabilization and we get explosively unstable behaviour. In the (unphysical) case of very large $\varkappa$-values, the whole marginal boundary lies in the area $\vartheta_T > 0$ of nonlinear destabilization, except for extremely close wall position ($x_w^{-2}$ very close to one).

Additional information concerning the magnitude of $\vartheta_T$ is presented in Figs. 4–7 which show $\tilde{\vartheta}_T = \vartheta_T \varrho_0 r_0^7 / B_{0\varphi}^2$ as a function of $x_w^{-2}$ for several values of $\varkappa$ and as a function of $\varkappa$ for several values of $x_w^{-2}$ respectively. It is seen from Fig. 7 ($x_w = 1.15$) that $\vartheta_T$ approaches zero as $\varkappa \to 0$, and the same was found for all other values of $x_w$. This means that $\varkappa = 0$ does not only limit the region of applicability of the nonlinear theory but is also the boundary of the adjacent regions of positive or negative values $\vartheta_T$. Fig. 8 shows the derivative $d\tilde{\vartheta}_T/d\varkappa$ for $\varkappa = 0$ which indicates a change of the nonlinear behaviour across the boundary $\varkappa = 0$. By the argument of continuity one is tempted to infer from this nonlinear stabilization of the $m = n = 1$ mode also near the marginal boundary $q = -1$ at values $x_w^{-2}$ where the curve $\varkappa = 0$ comes very close to this ($x_w^{-2}$ close to 1, i.e. stabilizing wall very close to the plasma). Finally, Fig. 9 shows the regions of nonlinear stabilization and destabilization in a $\varkappa - x_w$-plane.

In the limit of very large aspect ratio (small $\varkappa$) which allows for considerable simplifications (tokamak scaling) we can compare our results with nonlinear results obtained by other authors. In [6] and [7] the wall distance for which nonlinear stabilization becomes possible was found to be $x_w \gtrsim 1.43$ and $x_w \gtrsim 1.54$ respectively in this case. According to Figs. 3 and 8 we get $x_w^{-2}$ slightly above 0.41 or $x_w$ slightly below 1.56 as the critical wall distance at which the nonlinear behaviour turns over. The slight differences in these results are due to slight differences in the physical model, in the mathematical method and due to approximations introduced in [6] and [7].

Let us conclude with a few remarks on the spatial mode structure. Generally, we did not find any significant changes of it in the transition from nonlinear stabilization to destabilization. Figures 10a, b, c show three typical cases for the nonlinear correction $\varepsilon^2 \xi_2$ to the linear plasma shift $\varepsilon \xi_1$ which in a $z$ = const plane looks almost like a rigit shift. Finally, Figs. 11a and b show typical changes in the shape of the plasma boundary which the latter undergoes during the nonlinear oscillation in the regime of nonlinear stabilization. In a) the "orbit" encircles both bifurcating equilibria $A_0$ and $-A_0$, in b) only one of them.

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